Metro - Homework 12

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Course: 104142 - Metric and Topological Spaces

Question 1

(1.a) let X be T_1 space

Claim 1.a.1 every $Y \subset X$ is also T_1

Proof. let $Y \subset X$, take $x \neq y \in Y$, since $x,\ y \in X$, then exists $U \subset X$ open such that:

$$x \in U,\ y \notin U$$

by definition, hence, we have:

$$x \in U \cap Y\ ,\ \ y \notin U \cap Y$$

therefore, we have $V = U \cap Y$ the desired open nghbr, therefore Y is T_1 as required. $\square$

(1.b)

Claim 1.b.1 if X is normal, and $A \subset X$ a closed set, then A is normal

Proof. let $B_1,\ B_2 \subset A$ be both disjoint closed sets, since A is closed, then we also have $B_1,\ B_2$ closed in X, therefore, since X is normal, by definition, exists $V_1, V_2 \subset X$ disjoint open such that:

$$B_1 \subset V_1,\ B_2 \subset V_2$$

Consequently,

$$B_1 = B_1 \cap A \subset V_1 \cap A = U_1 \in \tau_A$$

$$B_2 = B_2 \cap A = V_2 \cap A = U_2 \in \tau_A$$

Hence,

$$B_1 \subset U_1,\ B_2 \subset U_2$$

$$U_1 \cap U_2 = \emptyset$$

Therefore, A is normal by definition. $\square$

Question 2

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let X be a topological space and $A \subset X$ closed set.

define the following relation:

$$x \sim y \iff x = y \ \vee \ \{x, y\} \subset A$$

Claim 2.1 if X is T_3 then $X/_\sim$ is T_2

Proof. let $[x] \neq [y] \in X/_\sim$, then by the definition of the relation we have:

$$(x \neq y) \ \wedge \ (\{x, y\} \not\subset A)$$

therefore, since $x \neq y$, and X is T_2, exists V_x, V_y disjoint open nghbrs of x, y respectively. the problem of taking these sets immediately to $X/_\sim$ is that they might intersect A and then they are not disjoint in $\tau_{X/_\sim}$. hence, assume without loss of generality, $x \notin A$. by the regularity of X, for $x \notin A$, exists $x \in U_x$, open such that:

$$U_x \cap A = \emptyset$$

and T_x open such that:

$$A \subset T_x$$

such that:

$$T_x \cap U_x = \emptyset$$

Therefore, we define:

$$W_x = V_x \cap U_x$$

and:

$$W_y = \begin{cases} T_x & y \in A \\ V_y \cap U_y & o.w \end{cases}$$

where in the case $y \notin A$ we take U_y to be the separating set from A.

Consequently, we have that both $W_y,\ W_x$ non empty since they contain $x,\ y$ respectively, and disjoint since W_x is disjoint to both T_x and V_y. hence, we take:

$$Z_x = \pi(W_x) = \{[z] : z \in W_x\}$$

$$Z_y = \pi(W_y) = \{[z] : z \in W_y\}$$

Since $W_x \cap A = \emptyset$, then $[A] \notin Z_x$. therefore, Z_x is open in $X/_\sim$ since:

$$\pi^{-1}(\pi(W_x)) = Z_x = \Big\{\{z\} : z \in W_x\Big\}$$

and W_y is open since, if $y \notin A$, then we have the same case as Z_x and hence open. otherwise, if $y \in A$ then:

$$\pi^{-1}(Z_y) = \pi^{-1}(\pi(W_y)) = W_y \cup A = (T_x \cup A) = T_x$$

which is open.

We have $Z_x,\ Z_y$ disjoint, since otherwise:

$$[z] \in Z_x \cap Z_y$$

then, since $[z] \neq [A]$ then

$$z \in W_x \cap W_y$$

In contradiction to their disjoint.

hence, we have:

$$[x] \in Z_x$$

$$[y] \in Z_y$$

and they are open and disjoint, therefore, $X/_\sim$ is T_2 as required. $\square$

Question 3

Claim 3.1 let X be regular space and second countable then X is normal.

Proof. let $\{B_n\}_{n=1}^{\infty}$ be a countable basis for X, take $A,\ B \subset X$ both closed disjoint. for each $a \in A$, we have $a \notin B$, therefore, by the regularity of X, there exists disjoint Q_a, W_a open such that:

$$a \in Q_a,\ B \subset W_a$$

hence, we have:

$$A \subset \bigcup_{a \in A} Q_a$$

since X is second countable, there exists $f\colon A \to \mathbb{N}$ such that we have:

$$a \in B_{f(a)} \subset Q_a$$

therefore, we replace the indexing of A with $\{x_n\}_{n=1}^{\infty} \subset f(A)$, and define $U_n = B_{x_n}$ hence we have:

$$A \subset \bigcup_{n=1}^{\infty} U_n$$

and for each n, pick $a_n \in f^{-1}(x_n)$ and define:

$$B \subset W_{a_n} = V_n$$

where a_n is a representative of such $f^{-1}(n)$'s. therefore, we obtain that:

$$U_n \subset Q_{a_n} \implies U_n \cap V_n = \emptyset$$

now define the following:

$$U_n' = U_n \setminus \bigcup_{k=1}^{n} \overline{V_k}$$

$$V_n' = V_n \setminus \bigcup_{k=1}^{n} \overline{U_k}$$

which are open sets, since they are open sets minus closed ones.

and we have

$$\overline{U_n} \cap B = \emptyset$$

because otherwise, take $x \in \overline{U_n} \cap B$, then we have:

$$x \in B \implies x \in V_n$$

V_n is an open set, therefore, exists P open nghbr of x such that:

$$x \in P \subset V_n$$

and,

$$x \in \overline{U_n} \implies P \cap U_n \neq \emptyset$$

Therefore,

$$U_n \cap V_n \neq \emptyset$$

in contradiction.

and we have

$$V_n' \cap A = \emptyset$$

hence, U_n' does not remove elements from A, meaning we have

$$A \subset \bigcup U_n \implies A \subset \bigcup U_n'$$

$$B \subset \bigcup V_n \implies B \subset \bigcup V_n'$$

define:

$$U = \bigcup_{n=1}^{\infty} U_n' \ , \quad V = \bigcup_{n=1}^{\infty} V_n'$$

both of which are open, because they are union of open sets. therefore,

$$U \cap V = \emptyset$$

because, otherwise, $\exists x$ such that:

$$x \in U \implies \exists i \;\; x \in U_i'$$

$$x \in V \implies \exists j \;\; x \in V_j'$$

if without loss of generatlity $i < j$, then we have:

$$x \in V_j' \implies x \in V_j \setminus \bigcup_{k=1}^{j} \overline{U_k}$$

In particular, we must have:

$$x \notin \overline{U_i}$$

xz which is a contradiction to $x \in U_i'$.

Consequently, we have

$$A \subset U, \ B \subset V$$

both open and disjoint as required.

$\square$