Metro - Homework 9

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Question 1

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let Y be compact topological space.

(1.a)

Claim 1.a.1 let V be open in $X \times Y$ with $\{x\} \times Y \subset V$, then exists $x \in W \subset X$ open such that:

$$W \times Y \subset V$$

Proof. by the assumption, $\forall y \in Y \quad (x, y) \in V$, with V open, hence for each $y \in Y$ there exists:

$$(x, y) \in A_y \times B_y \subset V$$

with A_y open in X and B_y open in Y. therefore, we have $\mathcal{U} = \{B_y\}$ is an open cover of Y, since Y is compact, then exists $|J| < \infty$ for finite sub cover of Y, hence we have:

$$\bigcup_{j \in J} B_j = Y$$

By definition we have:

$$\bigcup_{j \in J} A_j \times B_j \subset V$$

Hence, let:

$$W = \bigcap_{j \in J} A_j$$

is a finite intersection of open sets, therefore, W is open in X, and is not empty since $x \in W$, Moreover,

$$W \subset \bigcup_{j \in J} A_j$$

Hence, we have:

$$W \times Y \subset \bigcup_{j \in J} A_j \times B_j \subset V$$

As required. $\square$

(1.b)

Claim 1.b.1 the projection $\pi\colon X \times Y \to X$ is closed.

Proof. let V be closed set in $X \times Y$, we need to show that $\pi[V]$ is a closed set. define the following:

$$U = \pi[V]^c$$

and take $x \in U$, by definition, of x:

$$\forall y \in Y \quad (x, y) \notin V$$

since otherwise, $(x, y) \in V \implies x \in \pi[V]$, hence we have:

$$\{x\} \times Y \subset V^c$$

In which V^c is an open set, hence by (1.a.1), there exists $x \in W \subset X$ open such that:

$$W \times Y \subset V^c$$

Then we have:

$$W \cap \pi[V] = \emptyset$$

Since otherwise, if $w \in W \cap \pi[V]$, then exists $y \in Y$ such that:

$$(w, y) \in V$$

Which implies $(w, y) \in V \cap V^c$ in contradiction. As a result:

$$x \in W \subset \pi[V]^c$$

Hence, $\pi[V]^c$ is open, which implies $\pi[V]$ is closed as required. $\square$

(1.c)

Claim 1.c.1 let Y be hausdorf, let $f\colon X \to Y$ be function, then:

$$\text{f continuous} \iff G_f \text{ is closed}$$

Proof. Assuming f is continuous, we'll show that $(G_f)^c$ is open, let $(x, y) \in G_f^c$, then by definition we have:

$$z = f(x) \neq y$$

Since Y is hausdorf, there exists $V_z,\ V_y$ both open such that:

$$V_z \cap V_y = \emptyset$$

Since f is continuous, then $f^{-1}[V_z]$ is also open. define:

$$W = f^{-1}[V_z] \times V_y$$

Which is made up of open sets, hence open in $X \times Y$.

Assuming for purpose of contradiction that $W \cap G_f \neq \emptyset$, therefore:

$$\exists (x', y') \in W \ \ x' \in f^{-1}[V_z],\ y' \in V_y$$

$$(x', y') \in G_f \implies f(x') = y'$$

But, since $x' \in f^{-1}[V_z] \implies f(x') \in V_z$, then we have $y' = f(x') \in V_z$, which implies:

$$y' \in V_z,\ y' \in V_y$$

In contradiction to $V_z \cap V_y = \emptyset$, As a result:

$$(x, y) \in W \subset (G_f)^c$$

Hence, G_f is closed as required.

For the other direction, assume G_f is closed, let $V \subset Y$ be **closed** set, we'll show that $f^{-1}[V]$ is closed. first, we can see that:

$$f^{-1}[V] = \pi_X(G_f \cap (X \times V))$$

Since V is closed, then $X \times V$ is closed, and by the assumption, G_f is closed, hence finite intersection of closed, we have:

$$W = G_f \cap (X \times V)$$

is also a closed set, by the previous section (1.b.1) we have that $f^{-1}[V] = \pi_X[W]$ is also a closed set as required. $\square$

(1.d) let $X = \mathbb{S}^1$, $Y = [0,1)$, which has Y hausdorf and not compact since $Y \subset \mathbb{R}$ is not closed. define $\varphi\colon Y \to X$ as the following:

$$\varphi(x) = e^{2\pi i x}$$

we showed in class that φ is continuous and bijective, hence we define:

$$f\colon X \to Y$$

$$x \mapsto \varphi^{-1}(x)$$

In which f is not continuous, since if otherwise, we would have:

$$1 = \lim_{n\to\infty} 1 - \frac{1}{n} = \lim_{n\to\infty} f(e^{2\pi i(1-\frac{1}{n})}) = f(e^{2\pi i}) = 0$$

In contradiction, hence f is not continuous.

To show that G_f is closed, we take $(x_n, y_n) \subset G_f \subset \mathbb{S}^1 \times [0,1)$ a convergent sequence, hence we have:

$$\lim_{n\to\infty} x_n \in X$$

Since $(x_n, y_n) \in G_f$, then we have,

$$y_n = f(x_n) \implies f^{-1}(y_n) = x_n \implies \varphi(y_n) = x_n$$

Since φ is continuous, then:

$$\varphi(y) = \lim_{n\to\infty} \varphi(y_n) = \lim_{n\to\infty} x_n = x$$

Hence,

$$y = f^{-1}(x) \in Y$$

Consequently, $(x, y) \in G_f$ as required.

Question 2

Claim 2.1 let (X, d) be metric space, $A,\ B \subset X$ closed and disjoint, A compact, then:

$$0 < d(A, B)$$

Lemma 2.1 $\forall x \in A \ \ 0 < d(x, B)$

Proof. define $\delta_j = \dfrac{d(j, B)}{2}$ such that distance to a set is the standard infimum definition.

let:

$$U_a = B(a, \delta_a)$$

Hence, we have $\mathcal{U} = \{U_a \cap A\}$ is an open cover of A. since A is compact, then exists $|J| < \infty$ for finite subcover, hence we take:

$$\delta = \min_{j \in J}\{\delta_j\}$$

which is well defined since J is finite. let $y \in A$, then we have:

$$y \in B(j, \delta_j) \implies d(y, j) \leq \delta_j$$

for some j in J. By the triangle inequality we have:

$$d(j, B) \leq d(j, y) + d(y, B)$$

Consequently,

$$\delta_j = 2\delta_j - \delta_j \leq d(j, B) - d(j, y) \leq d(y, B)$$

For each $y \in A$, hence we have by definition of inf:

$$\delta \leq d(A, B)$$

As required.

$\square$