Metro - Topologist Sine Curve

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Question 1

Define the following sets.

$$G = \{(x, \sin \tfrac{1}{x}) : x > 0\}$$

$$T = \{(0, y) : -1 \leq y \leq 1\}$$

$$A = G \cup T$$

we want to show that A is connected. first

Lemma 1 $X = (0, 1]$ with the standard topology is connected

Proof. define the following sets:

$$A_n = \left[\frac{1}{n},\ 1\right]$$

these sets are connected for each n by a basic theorem on $\mathbb{R}$, and they are not disjoint since they all contain 1, therefore, by the union lemma we have:

$$\bigcup_{n=1}^{\infty} = (0, 1]$$

connected. □

Lemma 2 G is connected

Proof. define the following function $f\colon (0, 1] \to \mathbb{R}^2$:

$$f(x) = (x, \sin \tfrac{1}{x})$$

which is a continuous function on $\mathbb{R}^2$ since it's component wise continuous ($x > 0$), therefore, by a theorem, continuous image of a connected space is connected, hence:

$$G = f((0, 1])$$

is connected. □

Claim 1 A is connected.

Proof. first it's clear that for each $-1 \leq y \leq 1$ the point $(0, y)$ is a limit point for A since take any nghbr of $(0, y)$, say $B((0, y), \delta)$ and take,

$$x_n = \frac{1}{\arcsin y + 2\pi n}$$

since we have $x_n \xrightarrow{n\to\infty} 0$, then we pick n such that:

$$|x_n| < \delta$$

Therefore, we have

$$\sin\left(\frac{1}{x_n}\right) = \sin(\arcsin(y) + 2\pi n) = y$$

hence,

$$(x_n, y) \in G$$

and we have:

$$d((0, y), (x_n, y)) = |x_n| < \delta$$

Therefore,

$$(x_n, y) \in B((0, y), \delta)$$

as required.

therefore, we have that:

$$T \subset \overline{G}$$

hence we have:

$$G \subset G \cup T \subset \overline{G}$$

since G is connected, then by a theorem:

$$A = G \cup T$$

is connected, as required. $\square$

Claim 2 $X = G \cup \{(0,0)\}$ is not path connected

Proof. the idea here is to use "last exist time" for a continuous path, meaning, if a path were to exist, it cannot keep existing each nghbr of the point $(0,0)$, meaning it must settle down at some point, since it's continuous, therefore, we formalize this idea in the following:

assume for contradiction exsits a path connecting $(0,0)$ and $(\frac{1}{\pi}, 0)$ (or any point for that matter with $x > 0$)

Then exists $\gamma\colon [0,1] \to X$ with:

$$\gamma(0) = (0,0),\ \gamma(1) = (\tfrac{1}{\pi}, 0)$$

such that:

$$\gamma(t) = \big(\gamma_1(t),\ \gamma_2(t)\big)$$

since by definition, $\gamma \in X$, then:

$$\gamma_2(t) = \sin\left(\frac{1}{\gamma_1(t)}\right)$$

for $\gamma_1(t) \neq 0$, and $\gamma(t) = \vec{0}$ otherwise. Define

$$t_0 = \inf\{t : \gamma(t) \notin B((0,0), \delta)\}$$

for some $0 < \delta < 1$.

Assume for contradiction $t_0 = 0$. then γ is not continuous, since for each "small" nghbr $(\epsilon < \delta)$ exists t_1 with $\gamma(t_1)$ outside of this nghbr which is a contradiction to existence of the limit.

As a result, $0 < t_0$. by the assumption we have:

$$\vec{0} \neq \gamma(t_0)$$

hence, $\gamma_1(t_0) > 0$.

define the following:

$$x_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$$

since $x_n \xrightarrow{n\to\infty} 0$ then we can pick n such that:

$$0 < x_n < \gamma_1(t_0)$$

now since $\gamma_1 \colon [0,1] \to \mathbb{R}$ is a continuous function with:

$$\gamma_1(0) = 0 < x_n < \gamma_1(t_0)$$

by the IVT theorem, exists $0 < t_1 < t_0$ such that:

$$\gamma_1(t_1) = x_n$$

now since $t_1 < t_0$ we must have:

$$\gamma(t_1) \in B((0,0),\delta)$$

but, by the choice of x_n, we have:

$$\gamma(t_1) = \big(\gamma_1(t_1), \gamma_2(t_1)\big) = (x_n, \gamma_2(t_1))$$

since $\gamma(t_1) \in G$, then we have:

$$\gamma_2(t_1) = \sin\left(\frac{1}{x_n}\right) = 1$$

which is clearly outside the ball, since,

$$\delta < 1 \leq d((x_n, 1), (0,0))$$

therefore, a contradiction.

Hence, X is not path connected as required. $\square$