Metro - Noetherian and Irreducible spaces

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Question 1

Exercise 1.a.1 Is the space $X = \mathbb{R}$ with the standard topology a noetherian space?

Solution. No, $\mathbb{R}$ is not noetherian, take the following set:

$$F_n = \left[0,\ \frac{1}{n}\right]$$

Assuming $\mathbb{R}$ is noetherian we'll have for some $n_1 \neq n_2$:

$$F_{n_1} = [0, \tfrac{1}{n_1}] = [0, \tfrac{1}{n_2}] = F_{n_2}$$

Which implies:

$$n_1 = n_2$$

In contradiction. $\square$

Claim 1.b.1 let X be noetherian space, then $Y \subset X$ is also noetherian.

Proof. let $\{F'_i\}$ be family of decending closed sets in Y. then we have for each F'_i:

$$F'_i = F_i \cap Y$$

where F_i is closed in X, define the following set:

$$A_n = \bigcap_{i=1}^{n} F_i$$

Which is finite intersection of closed sets hence closed in X, and it is decending since:

$$A_n \supset A_{n+1}$$

Hence, by the assumption, exists $n_0 \in \mathbb{N}$:

$$\forall n \geq n_0 \;\; A_n = A_{n_0} \tag{1.1}$$

Now, by the definition of A_n we have:

$$A_n \cap Y = \bigcap_{i=1}^{n} F_i \cap Y = \bigcap_{i=1}^{n} F'_i = F'_n$$

Hence, by (1.1),

$$\forall n \geq n_0 \;\; F'_n = A_n \cap Y = A_{n_0} \cap Y = F'_{n_0}$$

As required. □

Claim 1.c.1

$$X \text{ is noetherian} \iff \forall U \in \tau \;\; U \text{ is compact}$$

Proof. assume X is noetherian, let U be an open set, by (1.b.1) U is also noetherian. let $\{U_i\}_{i\in I}$ be an open cover of U, assuming for contradiction U is not compact, then for any finite union:

$$U_1 \cup \cdots \cup U_{n_0} \neq U$$

therefore, we construct the following sequence by induction:

Define, $V_1 = U_1$, since $V_1 \neq U$, then exists $x_1 \in V_1^c$, now, since $\{U_i\}_{i\in I}$ is an open cover, and $x_1 \notin U_1$, then exists U_2 such that $x_1 \in U_2$, define $V_2 = U_1 \cup U_2$, since U is not compact, then $V_2 \neq U$, therefore, exists $x_2 \in V_2^c$ and again, for such x_2 exists U_3. continuing in induction we reach that:

$$V_n = U_1 \cup \cdots \cup U_n \;, \quad x_n \notin V_n$$

it's clear that V_n is an accending sequence of open sets, hence, define $F_n = (V_n)^c$, which is a decending sequence of closed sets, since U is noetherian, then exists n_0 such that:

$$\forall n \geq n_0 \;\; F_{n_0} = F_n$$

In particular,

$$(U_1 \cup \cdots \cup U_{n_0} \cup U_{n_0+1})^c = F_{n_0+1} = F_{n_0} = (U_1 \cup \cdots \cup U_{n_0})^c$$

which implies:

$$U_{n_0+1} \subset V_{n_0}$$

but we picked $x_{n_0} \in U_{n_0+1}$ such that $x_{n_0} \notin V_{n_0}$ which is a contradiction. hence, U is compact.

For the other direction, let $\{F_i\}$ be sequence of decending closed sets. define the following:

$$V = \bigcap F_i$$

Which is an arbitrary intersection of closed sets hence closed, then since $V^c \subset X$ is open, since:

$$V^c = \bigcup F_i^c$$

Therefore, $\{F_i\}$ is an open cover of V^c, by the assumption, exists a finite cover such that for $|J| < \infty$:

$$V^c = \bigcup_{j \in J} F_j^c$$

Consequently,

$$V = \bigcap_{j \in J} F_j$$

Since the sequence is of decending, then take $n_0 = \max\{J\}$ we have:

$$\bigcap^{\infty} F_i = V = F_{n_0}$$

Meaning,

$$\forall n \in \mathbb{N} \quad F_{n_0} \subseteq F_n$$

And since, $\forall n \geq n_0$ we have:

$$F_n \subseteq F_{n_0}$$

Then we have,

$$\forall n \geq n_0 \quad F_n = F_{n_0}$$

Hence X is noetherian as required. $\square$

Question 2

Claim 2.a.1 the following are equivalent

1. X is irreducible
2. $\forall U, V \in \tau \setminus \{\emptyset\} \;\; U \cap V \neq \emptyset$
3. $\forall U \in \tau \setminus \{\emptyset\} \;\; \bar{U} = X$

Proof. i'll prove the (1 2 3) cycle. each item is the assumed argument (so the first 1 is assuming 1 holds)

1. let U, V be non empty open sets, assuming $U \cap V = \emptyset$ for contradiction, then we have:
$$U^c \cup V^c = \emptyset^c = X$$
In contradiction to X irreducible.

2. let U be a non empty open set, let $x \in X$, and let V be an open nghbr of x, by the assumption, $U \cap V \neq \emptyset$, hence $x \in \bar{U}$ by definition.

3. let A, B be non empty closed sets, by the assumption $\overline{A^c} = X$, in particular, for $x \in B^c$ an open nghbr, we must have:
$$A^c \cap B^c \neq \emptyset$$
In particular,
$$A \cup B \neq X$$
Hence, X is irreducible. consequently, we showed $(1) \implies (2) \implies (3) \implies (1)$ which proves the equivalence, as required.

$\square$

Claim 2.b.1 if X is irreducible then X is connected.

Proof. let X be irreducible, assuming for contradiction X is not connected, then exists U, V disjoint open such that $U \cup V = X$. by (1.c.1) we must have:

$$U \cap V \neq \emptyset$$

In contradiction, hence, X is connected. $\square$

The converse is not true, take $X = [0, 1]$ which is a connected subset of $\mathbb{R}$, take $U = (\frac{1}{4}, \frac{1}{3})$, $V = (\frac{1}{2}, \frac{2}{3})$, which are open disjoint, but by (2.a.1) we must have $V \cap U \neq \emptyset$, in contradiction.

Lemma 2.c.1 let X be a space where $\forall x \in X \ \ \{x\}$ is clopen then $\tau_X = \mathbb{P}(X)$

Proof. let $A \subset X$, we want to show that $A \in \tau$. since we have:

$$A = \bigcup_{a \in A} \{a\}$$

where each $\{a\}$ is open set, then union of arbitrary open sets is open, hence $A \in \tau$, as required. $\square$

Claim 2.c.1 if X is noetherian, hausdorff then X is finite and $\tau = \mathbb{P}(X)$

Proof. let $x \in X$, since X is hausdorff, then $\{x\}$ is closed.

take $A = \{x\}^c$ which is open, and by (1.c.1) A is compact, hence by a theorem a compact set in hausdorff is closed. therefore,

$$\{x\}^c \text{ closed} \implies \{x\} \text{ open}$$

and by (2.c.1) $\tau = \mathbb{P}(X)$ for the first part.

now since A is compact, then we take:

$$U_y = \{y\}$$

for each $y \neq x$ as an open cover of A, then exists a finite subcover $|J| < \infty$, such that:

$$A = \bigcup_{j \in J} U_j$$

in which by definition, we have $J \subset X$, if X is not finite, then we can pick $z \in (J \cup \{x\})^c$.

since $z \in A$, we have:

$$z \in U_{j_0}$$

for some $j_0 \in J$, but this leads to $\{z\} = \{j_0\}$ with $z \neq j_0$ in contradiction, hence X is finite. $\square$

Question 3

Let X be a noetherian space, let $K = \{A : A^c \in \tau,\ A \text{ irreducible}\}$, define

$$\mathcal{S} = \{A : A^c \in \tau, \quad \nexists\ \{K_k\}_{k=1}^n \subset K \quad \bigcup_{j=1}^n K_j = A\}$$

which is the set of all closed sets in X that cannot be written as a finite union of closed irreducible sets.

Assuming for contradiction $\mathcal{S} \neq \emptyset$

Claim 3.a.1 exists a minimal element in $\mathcal{S}$ by the set inclusion order.

Proof. assume otherwise, and pick $A_1 \in \mathcal{S}$, which exists since $\mathcal{S} \neq \emptyset$, since A_1 is not minimal, then there exists $A_2 \in \mathcal{S}$ such that:

$$A_2 \subsetneq A_1$$

Since, A_2 is not minimal, then exists $A_3 \subsetneq A_2$. we build this sequence in induction and get:

$$A_{n+1} \subsetneq A_n$$

decending sequence of closed sets, since X is noetherian, then exists n_0 such that:

$$\forall n \geq n_0 \ \ A_n = A_{n_0}$$

In particular, $A_{n_0+1} = A_{n_0}$ in contradiction to $A_{n_0+1} \subsetneq A_{n_0}$. therefore, exists a minimal element in $\mathcal{S}$. $\square$

Claim 3.a.2 $S = \emptyset$

Proof. by (3.a.1), S has a minimum, therefore, define:

$$Y = \min\{S\}$$

it's clear that $K \bigcap \mathcal{S} = \emptyset$, because every set in K satisfies the finite union trivally.

Hence, $Y \notin K$, therefore, Y is not irreducible, meaning, exists non empty A, B closed such that:

$$A \cup B = Y$$

Since $A \subset Y,\ B \subset Y$, they cannot be in $\mathcal{S}$, since otherwise if $\mathcal{S} \cap \{A, B\} \neq \emptyset$, from the minimality of Y we have $Y \in \{A, B\}$ which is a contradiction to them being proper.

Hence, A, B are finitly reducable, meaning, exists:

$$A = K_1 \cup ... \cup K_n,\ B = K'_1 \cup ...K'_m$$

such that:

$$K_i \in K\ ,\ \ K'_i \in K$$

Therefore, we have:

$$Y = K_1 \cup ... \cup K_n \cup K'_1 \cup ...K'_m$$

which is a finite union of irreducible sets in K, therefore, $Y \notin S$, which is a contradiction to $Y \in S$. therefore, $S = \emptyset$ $\square$

Claim 3.b.1 X can be written uniquely up to order as

$$X = X_1 \cup ... \cup X_n$$

where each X_i is irreducible and $\forall i < j \;\; X_i \not\subset X_j$

Proof. by (3.a.2) we showed $\mathcal{S} = \emptyset$, consequently, $X \notin \mathcal{S}$, hence, by definition, there exists $\{X_k\}_{k=1}^n \subset K$ such that:

$$X = X_1 \cup ... \cup X_n$$

define the following set:

$$L = \{n : \{X_k\}_{k=1}^n \subset K \;,\;\; X = \bigcup X_k\}$$

which is the set of lengthes of the irreducible sequence that make up X. since $\emptyset \neq L \subset \mathbb{N}$ and it's well ordered, then exists $n = \min\{L\}$. to show that $\forall i < j \;\; X_i \not\subseteq X_j$, assume otherwise, then we have:

$$X_i \cup X_j = X_j$$

Therefore, exists an irreducible sequence with length less than n, in contradiction to n minimal.

To show uniqueness, let $\{Y_k\}_{k=1}^n$ be irreducible closed such that:

$$X = Y_1 \cup ... \cup Y_n$$

Hence, we have:

$$X_i = (Y_1 \cap X_i) \cup ... \cup (Y_n \cap X_i)$$

Since X_i is irreducible, then exists j such that:

$$X_i = (Y_j \cap X_i)$$

Consequently, $X_i \subseteq Y_j$.

In the same spirit for Y_j, we have:

$$Y_j = (X_1 \cap Y_j) \cup ... \cup (X_n \cap Y_j)$$

Hence, exists k such that:

$$Y_j = (X_k \cap Y_j)$$

Therefore, we have:

$$X_i \subseteq Y_j \subseteq X_k$$

we showed above that we must have $i = k$, therefore, $X_i \subseteq Y_j \subseteq X_k = X_i$.

Hence, we showed:

$$\forall 1 \leq i \leq n \;\; \exists! j \;\; X_i = Y_j$$

Therefore, $\{X_k\}_{k=1}^n = \{Y_k\}_{k=1}^n$ $\square$

Question 4

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let $\mathbb{P} \subset \mathbb{N}$ be the subset of the prime numbers, define $X = \mathbb{P} \cup \{0\}$, and define:

$$\mathcal{F} = \{X\} \cup \{A \subset \mathbb{P} : |A| < \infty\}$$

be the set of closed sets, define τ as:

$$U \in \tau \iff U^c \in \mathcal{F}$$

Lemma 4.a.1 if U is non empty open in τ then $0 \in U$ and $|U^c| < \infty$

Proof. U is open, then $U^c \in \mathcal{F}$, since $U \neq \emptyset$, then $U^c \neq X$, hence, we must have:

$$U^c \subset \mathbb{P},\ |U^c| < \infty$$

Hence, since 0 is not prime, then $0 \notin U^c$, meaning $0 \in U$. $\square$

Lemma 4.a.2 if $|K^c| < \infty$ and $0 \notin K^c$, then K is open.

Proof. since $0 \notin K^c$, then $K^c \subset \mathbb{P}$, and by the assumption, we have $|K^c| < \infty$ hence $K \in \tau$ by definition. $\square$

Claim 4.a.1 τ is a topology

1. since $X \in \mathcal{F}$ then $\emptyset \in \tau$, and since $\emptyset \subset \mathbb{P}$ and $|\emptyset| < \infty$ then $X \in \tau$

2. let $\{U\}$ be collection of not empty open sets (empty sets don't contribute to the union) by (4.a.1) we have $0 \in U$ and $|U^c| < \infty$, hence, for

$$K = \bigcup U \implies K^c = \bigcap U^c$$

Therefore, $0 \in K$ and K^c is intersection of finite sets hence $|K^c| < \infty$ and $0 \notin K^c$ then by (4.a.2), K is open.

3. let $\{U_k\}_{k=1}^{n}$ be finite collection of non empty open sets, again by (4.a.1) we have $0 \in U$ and $|U^c| < \infty$, hence for:

$$K = \bigcap_{k=1}^{n} U_k \implies K^c = \bigcup_{k=1}^{n} U_k^c$$

Hence, $0 \notin K^c$ and K^c is finite union of finite sets hence finite, therefore, by (4.a.2) K is open.

Claim 4.b.1 (X, τ) is noetherian space.

Proof. let $\{F_n\}_{n=1}^{\infty} \subset X$ be descending sequence of closed sets.

If all $F_n = X$, then the claim is trivial, otherwise, exists $F_{n_0} \neq X$, therefore, we consider the sequence to start from there.

Since F_n is closed and is not X, then by (4.a.1) we have $|F_n| < \infty$, define:

$$x_n = |F_n|$$

By the assumption we have:

$$0 \leq x_{n+1} \leq x_n$$

define the following:

$$V = \{x_n : n \in \mathbb{N}\}$$

which is a finite set since $V \subset \{0, ..., x_1\} \subset \mathbb{N}$, hence exists $x_{n_0} = \min\{V\}$ such that:

$$\forall n \in \mathbb{N} \;\; x_{n_0} \leq x_n$$

consequently, by the monotonicity of x_n we have:

$$\forall n_0 \leq n \;\; x_n \leq x_{n_0}$$

as a result, we have:

$$\forall n_0 \leq n \;\; x_{n_0} = x_n$$

hence, we have $F_n \subseteq F_{n_0}$ and $|F_{n_0}| = |F_n|$ and both are finite, therefore, we get:

$$\forall n \geq n_0 \;\; F_{n_0} = F_n$$

As required. $\square$

Claim 4.c.1 X is irreducible

Proof. let U, V be open sets, by (4.a.1) $0 \in V,\ 0 \in U$, hence their intersection is not empty, therefore, by (2.a.1) X is irreducible. $\square$