Metro - Homework 11

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Question 1

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let (X, d) be a complete metric space, take $\mathcal{F} \subset C(X, \mathbb{R})$, assume:

$$\forall x \in X \ \ S_x = \sup\{|f(x)| : f \in \mathcal{F}\} < \infty$$

Claim 1 exists $U \subset X$ open, and $M \in \mathbb{R}$ such that:

$$\forall f \in \mathcal{F} \ \ \forall x \in U \ \ |f(x)| \leq M$$

Proof. define the following set:

$$A_n = \{x : \forall f \in \mathcal{F} \ \ |f(x)| \leq n\}$$

We'll show that A_n is closed, then use baire theorem on X and conclude that there must exists A_k with $(A_k)^\circ \neq \emptyset$.

let $x \in A_n^c$, then by definition, exists $f \in \mathcal{F}$ such that:

$$n < |f(x)|$$

Since f is continuous at x then we pick $\epsilon = |f(x)| - n > 0$, hence there exists $\delta > 0$ such that:

$$y \in B(x, \delta) \implies |f(x) - f(y)| < \epsilon = |f(x)| - n$$

Then by the triangle inq (reverse):

$$||f(x)| - |f(y)|| < |f(x)| - n$$

Consequently,

$$-(|f(x)| - n) < |f(x)| - |f(y)| \implies n < |f(y)|$$

Hence, we have:

$$B(x, \delta) \subset A_n^c$$

Therefore, A_n is closed.

let $x \in X$, since by the assumption $S_x < \infty$, then exists k such that $x \in A_k$ Hence, we have:

$$\bigcup_{n=0}^{\infty} A_n = X$$

Since X is complete, by baire theorem, if all A_n were to have an empty interior, then X would also have empty interior, in contradiction, hence, exists k such that $(A_k)^\circ \neq \emptyset$, meaning, exists $U \subset A_k$ open, such that

$$\forall x \in U \ \ \forall f \in \mathcal{F} \ \ |f(x)| \leq k$$

As required. $\square$

Question 2

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let $X = \mathbb{R}[x]$, define the norm:

$$\|\sum_{k=0}^{n} a_k x^k\| = \sqrt{\sum_{k=0}^{n} a_k^2}$$

Claim 1 $(X, \|\cdot\|)$ is not complete.

Proof. define the following sequence:

$$p_n(x) = \sum_{k=0}^{n} \frac{x^k}{\sqrt{k!}}$$

first, since (by infi) we know that:

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e$$

is a convergent series in $\mathbb{R}$. hence, it's cauchy, therefore, let $\epsilon > 0$, exists $N > 0$ such that:

$$\|p_n(x) - p_m(x)\|^2 = \|\sum_{k=n}^{m} \frac{x^k}{\sqrt{k!}}\|^2 = \sum_{k=n}^{m} \frac{1}{k!} < \epsilon$$

hence $\{p_n(x)\}_{n=1}^{\infty}$ is cauchy in $\mathbb{R}[x]$.

assuming for contradiction that it converges, let $q(x) \in \mathbb{R}[x]$, be it's limit:

$$\lim_{n\to\infty} \|p_n(x) - q(x)\| = 0 \tag{2.1}$$

Then, for $q(x) = \sum_{k=0}^{d} a_k x^k$ with $d = \deg(q)$, and $n > d$, we have:

$$\|p_n(x) - q(x)\|^2 = \underbrace{\sum_{k=0}^{d}(\frac{1}{\sqrt{k!}} - b_k)^2}_{M} + \underbrace{\sum_{k=d+1}^{n} \frac{1}{k!}}_{S_n}$$

Hence, we have,

$$\lim_{n\to\infty} S_n = e - \sum_{k=0}^{d} \frac{1}{k!} > 0$$

A strictly positive limit.

By 2.1, we have:

$$\lim_{n\to\infty} M + S_n = 0$$

Consequently,

$$\lim_{n\to\infty} S_n = -M \leq 0$$

Which is a contradiction. in particular,

$$\sum_{k=0}^{d} (\frac{1}{\sqrt{k!}} - b_k)^2 = -\sum_{k=d}^{\infty} \frac{1}{k!}$$

hence we have a finite non-negative sum equal a strictly negative sum which is a contradiction. hence no such $q(x)$ exist.

Therefore, $\mathbb{R}[x]$ is not complete. $\square$