Metro - Homework 10

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Question 1

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let (X, d) be metric space.

(1.a) let $X' = \{\{x_n\}_{n=1}^{\infty} \subset X : \{x_n\}_{n=1}^{\infty} \text{ is cauchy}\}$ with relation:

$$\{x_n\}_{n=1}^{\infty} \sim \{y_n\}_{n=1}^{\infty} \iff d(x_n, y_n) \xrightarrow{n\to\infty} 0$$

Claim 1 $\sim$ is an equivalence relation

Proof. let $\{x_n\}_{n=1}^{\infty} \in X'$

1. since d is metric, then $d(x_n, x_n) = 0$, hence $x_n \sim x_n$

2. let $y_n \in X'$, since d is metric, $d(x_n, y_n) = d(y_n, x_n)$ consequently,

$$x_n \sim y_n \iff 0 = \lim d(x_n, y_n) \iff \lim d(y_n, x_n) = 0 \iff y_n \sim x_n$$

3. let $\{z_n\} \in X'$ with $x_n \sim y_n,\ y_n \sim z_n$ then by the traingle inequality:

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) \xrightarrow{n\to\infty} 0$$

   Hence $x_n \sim z_n$

$\square$

(1.b) define $\tilde{X} = X/_{\sim}$ with $\tilde{d}\colon \tilde{X} \times \tilde{X} \to \mathbb{R}$:

$$\tilde{d}([x_n], [y_n]) = \lim_{n\to\infty} d(x_n, y_n)$$

Claim 1 $\tilde{d}$ is a metric

Proof. first we'll show that $\tilde{d}([x_n],[y_n])$ is defined on $\mathbb{R}$, let $z_n = d(x_n, y_n)$, which is a sequence in $\mathbb{R}$, then since both x_n and y_n are cauchy, we have:

$$|z_n - z_m| = |d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

hence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is cauchy sequence in $\mathbb{R}$ complete, hence z_n is convergent and the limit exists.

we'll show it's well defined, let $[x_n] = [x'_n],\ [y_n] = [y'_n]$, then we have:

$$\begin{aligned}\tilde{d}([x'_n],[y'_n]) &= \lim_{n\to\infty} d(x'_n, y'_n) \leq \lim_{n\to\infty} d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n) \\ &= 0 + \tilde{d}([x_n],[y_n]) + 0 = \tilde{d}([x_n],[y_n])\end{aligned}$$

hence, it's well defined. now to show it's a metric:

1. by the definition of $\sim$ we have

$$\tilde{d}([x_n],[y_n]) = 0 \iff \lim_{n\to\infty} d(x_n, y_n) = 0 \iff [x_n] = [y_n]$$

2. by the positiveness of d a metric we have $\tilde{d} > 0$

3. since d is symmetric then we also have $\tilde{d}$ symmetric

4. let $z_n \in X$ then we have:

$$d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n) \implies \lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} d(x_n, z_n) + d(z_n, y_n)$$
$$\tilde{d}([x_n],[y_n]) = \lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n) = \tilde{d}([x_n],[y_n]) + \tilde{d}([y_n],[z_n])$$

□

(1.c)

Claim 1 $(\tilde{X}, \tilde{d})$ is complete

first we construct y_n, let $\{\tilde{x}_n\} \subset \tilde{X}$ be cauchy sequence, then we define y_n as the following:

$$y_n = f(\tilde{x}_n)_{i_n}$$

where $f\colon \tilde{X} \to X^{\mathbb{N}}$ is a function that returns a representative of the class $[\tilde{x}_n]$, and since $f(\tilde{x}_n)$ is a cauchy sequence in X then we pick i_n to be the index such that we have, $\forall \alpha,\ \beta \geq i_n\ :$

$$d(f(\tilde{x}_n)_\alpha, f(\tilde{x}_n)_\beta) < \frac{1}{n}$$

Lemma 1 y_n is cauchy in X

Proof. let $\epsilon > 0$, take $N_1 > \dfrac{3}{\epsilon}$, then for each $n, m > N_1$ we have, by the definition of $\{y_n\}$:

$$\forall k > i_n\ \ d(y_n, f(\tilde{x}_n)_k) < \frac{1}{n} < \frac{1}{N_1} < \frac{1}{3}\epsilon$$
$$\forall k > i_m\ \ d(y_m, f(\tilde{x}_m)_k) < \frac{1}{3}\epsilon$$

Since $\{\tilde{x}_n\}$ is cauchy then exists N_2 such that for each $n, m > N_2$:

$$\tilde{d}(\tilde{x}_n, \tilde{x}_m) < \frac{1}{3}\epsilon$$

In particular,

$$\lim_{k\to\infty} d(f(\tilde{x}_n)_k, f(\tilde{x}_m)_k) < \frac{1}{3}\epsilon$$

Therefore, we take k such that we have:

$$d(f(\tilde{x}_n)_k, f(\tilde{x}_m)_k) < \frac{1}{3}\epsilon$$

Hence, for $N = \max\{N_1, N_2\}$ we have for each $n, m > N$

$$d(y_n, y_m) \leq d(y_n, f(\tilde{x}_n)_k) + d(f(\tilde{x}_n)_k, f(\tilde{x}_m)_k) + d(f(\tilde{x}_m)_k, y_m)$$

and by the previous claims we have:

$$d(y_n, y_m) < \epsilon$$

As required $\square$

Hence we have that $\{y_n\}_{n=1}^{\infty} \in X'$

Lemma 2

$$\lim_{n\to\infty} \tilde{x}_n = [y]$$

Proof. by the definition of $\tilde{d}$ we have:

$$\tilde{d}(\tilde{x}_n, [y]) = \lim_{k\to\infty} d(f(\tilde{x}_n)_k, y_k)$$

By the construction of $\{y_n\}$, let $\epsilon > 0$

$$d(f(\tilde{x}_n)_k, y_k) \leq \underbrace{d(f(\tilde{x}_n)_k, y_n)}_{\text{construction of } y_n} + \underbrace{d(y_n, y_k)}_{\text{is cauchy}}$$

Since for $\epsilon > 0$, for y_n cauchy exists N, take $n, k > \max\{N, \frac{1}{\epsilon}\}$, hence we have:

$$d(f(\tilde{x}_n)_k, y_n) < \frac{1}{k} = \epsilon,\ d(y_n, y_k) < \epsilon$$

Therefore,

$$\lim_{n\to\infty} \tilde{d}(\tilde{x}_n, [y]) = 0$$

As required □

Now for (1.c.1)

Proof. we have $\{\tilde{x}_n\}_{n=1}^{\infty}$ any cauchy sequence in $\tilde{X}$ convergent, hence $\tilde{X}$ is complete, as required. □

(1.d) define $i\colon X \to \tilde{X}$ as the following:

$$i(x) = [\{x_n\}_{n=1}^{\infty}]$$

where $\forall n \in \mathbb{N}\ \ x_n = x$

Claim 1 i is isometric

Proof. by definition of $\tilde{d}$ we have:

$$\tilde{d}(i(x), i(y)) = \lim_{k\to\infty} d(x_n, y_n) = \lim_{k\to\infty} d(x, y) = d(x, y)$$

hence i is isometric by definition. □

(1.e)

Claim 1 $i(X)$ is dense in $\tilde{X}$

Proof. let $\tilde{x} \in \tilde{X}$, we want to show that for every open nghbr $B(\tilde{x}, r)$ it intersects $i(X)$. since for $\tilde{x} = [\{x_n\}]$ we have $f(\tilde{x}) = \{x_n\}$ cauchy, then for $\epsilon = \frac{r}{2}$ exists $N > 0$ such that for all $n, m \geq N$

$$d(x_n, x_m) < \frac{r}{2}$$

define K as the following:

$$K = \sup_{N<n}\{d(x_n, x_N)\}$$

therefore, we have $K \leq \frac{r}{2}$, consequently, we pick $y_n = x_N$ constant sequence, we have

$$\tilde{d}(\tilde{x}, [y_n]) = \lim_{k\to\infty} d(f(\tilde{x})_k, y_k) \leq K \leq \frac{r}{2} < r$$

hence, $[y_n] \in B(\tilde{x}, r)$, and since $\{y_n\}_{n=1}^{\infty}$ is constant, then $[\{y_n\}] \in i(X)$, hence we have:

$$[\{y_n\}] \in B(\tilde{x}, r) \cap i(X)$$

As required. $\square$

Question 2

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the claim is **incorrect**, take $X = (0,1)$, , $Y = \mathbb{R}$.

Claim 1 X is homeomorphic to Y

Proof. we know the function $\tan\colon (\frac{-\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is bijective and continuous, moreover, it's inverse, $\arctan\colon (\frac{-\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is also continuous, hence we define, $f\colon (0,1) \to \mathbb{R}$ as the following:

$$f(x) = \tan(\varphi(x))$$

with $\varphi\colon (0,1) \to (\frac{-\pi}{2}, \frac{\pi}{2})$

$$\varphi(x) = \pi x - \frac{\pi}{2}$$

which is bijective continuous/open, therefore, f is bijective continuous/open as required. $\square$

$\mathbb{R}$ is complete, but $(0,1)$ is not, which can be shown by the example:

$$x_n = \frac{1}{n}$$

we have $x_n \xrightarrow{n\to\infty} 0$, therefore by theorem: $\{x_n\}$ is convergent (in $\mathbb{R}$) hence cauchy. but $0 \notin (0,1)$ therefore, $\{x_n\}$ is not convergent in $(0,1)$ therefore, it's not complete.