Metro - Baire category theorem

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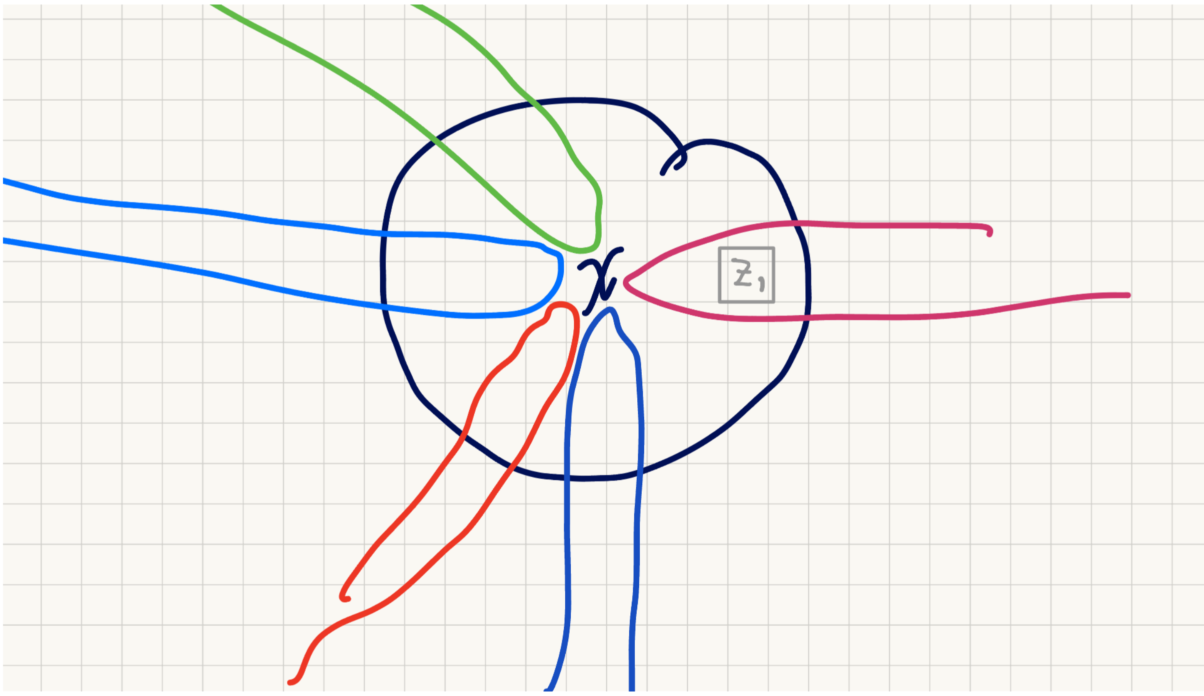
Claim 1 let X be complete metric space, $U_1,\ U_2,\ ...$ be open dense sets. then:

$$V = \bigcap_{n=1}^{\infty} U_n$$

is dense.

the problem in an "easy" proof, is that while it's true that $B(x,\epsilon) \cap U_i \neq \emptyset$ for each i, but it does not gaurantee that, $V \cap B(x,\epsilon) \neq \emptyset$, since we might have a case like:



in which the density (and openness) of each U_i solves, since we pick z_1 in $U_1 \cap B(x,\epsilon)$ like in the image and repeat the process which must lead to non empty intersection.

so we turn to the following

Lemma 1 let $x \in X,\ \epsilon > 0,\ z_0 = x,\ \delta_0 = \epsilon$. Then $\forall n \in \mathbb{N}$ exists $z_n \in X,\ \delta_n < \frac{1}{n}$ such that:

$$z_n \in \overline{B(z_{n-1}, \delta_n)} \subset U_n \cap B(z_{n-1}, \delta_{n-1})$$

Proof. Base case, Since U_1 is dense, then for $B(x,\epsilon)$ open we have:

$$U_1 \cap B(x,\epsilon) \neq \emptyset$$

take $y \in U_1 \cap B(x,\epsilon)$, since $U_1 \cap B(x,\epsilon)$ is open, then exists α such that:

$$B(y,\alpha) \subset U_1 \cap B(x,\epsilon)$$

take $\delta_1 = \dfrac{\min\{\alpha, 1\}}{2}$, therefore, we have:

$$y \in \overline{B(y, \delta_1)} \subset B(y, \alpha) \subset U_1 \cap B(x, \epsilon)$$

Therefore, set $z_1 = y$.

For general case, assume we have $z_1, ..., z_n$ constructed. then for z_{n+1} we do the same as the base step, with

$$U_{n+1} \cap B(z_n, \delta_n)$$

is open, and U_{n+1} is dense, so

$$z_{n+1} \in U_{n+1} \cap B(z_n, \delta_n) \neq \emptyset$$

and by the above construction method we take the appropriate $\delta_{n+1} < \frac{1}{n+1}$. □

Lemma 2 z_n as constructed above is cauchy

Proof. let $\epsilon > 0$, and take N such that $\dfrac{1}{N} < \epsilon$, and take $N < n < m$, by the above construction:

$$z_m \in U_{m-1} \cap B(z_{m-1}, \delta_{m-1})$$

By induction, we reach:

$$z_m \in U_n \cap B(z_n, \delta_n)$$

Hence:

$$d(z_m, z_n) < \delta_n$$

By the construction of δ_n we have:

$$\delta_n < \frac{1}{n} < \frac{1}{N} < \epsilon$$

Hence $\{z_n\}_{n=1}^{\infty}$ is cauchy. □

Lemma 3 z_n as constructed above converges to z and $\forall n \in \mathbb{N}\ \ z \in U_n$

Proof. first, since $\{z_n\}_{n=1}^{\infty}$ is cauchy in X complete, it converges to z, and it's unique. by the construction of z_n we have:

$$\forall n < k \ \ z_k \in \overline{B(z_n, \delta_n)} = K_n$$

Since $K_n \subset X$ is a closed set, and X is a complete metric space, then K_n is complete, then for $\{z_k\}_{k=n+1}^{\infty} \subset K_n$ a cauchy sequence, it converges in K_n, and by the uniqueness of the limit in each K_n, we have:

$$z \in K_n \subset U_n$$

unique as required. $\square$

Now returning to our main claim (0.1)

Proof. since we picked $x \in X,\ \epsilon > 0$ arbitrary, then showed that $z_n \xrightarrow{n\to\infty} z$ such that:

$$z \in \overline{B(z_1, \delta_1)} \subset B(x, \epsilon)$$

and be (0.3), also $\forall n \in \mathbb{N}\ \ z \in U_n$, then we have:

$$z \in \bigcap_{n=1}^{\infty} U_n \cap B(x, \epsilon)$$

Which shows that $\bigcap_{n=1}^{\infty} U_n$ is dense as required. $\square$