Injective entire functions

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Question 1

Lemma 1 if $f : \Omega \to \mathbb{C}$ is holomorphic non-constant with $f(0) = 0$ a zero of order $m \geq 2$, then f is not injective.

Proof. by theorem, there exists $r > 0$ such that:

$$f(z) = z^m h(z)$$

with $h(z) \neq 0$, for $z \in D_r(0)$. since $m \geq 2$, then we also have $f'(0) = 0$, but since f is not constant, f' is not identically zero,

therefore, exists $\epsilon > 0$ such that for $0 < |z| \leq \epsilon$:

$$f'(z) \neq 0 \wedge f(z) \neq 0$$

which isolates both function's zeros.

define $k = \min_{C_\epsilon(0)} |f|$ which exists on the compact circle, and non zero as assumed.

take $0 < |w| < k$, then define:

$$g(z) = -w$$

then we have:

$$|g(z)| = |w| < k \leq |f(z)|$$

on $C_\epsilon(0)$.

therefore, by rouche's theorem f and $f + g$ have the same number of zeros in $D_\epsilon(0)$.

since f has a zero of order more than 2, then $h(z) = f(z) - w$ has zeros counted with multiplicity at total of m, take $z_0 \in \mathbb{C}$ such that:

$$h(z_0) = 0 \implies f(z_0) = w$$

therefore, z_0 is a zero of h, but since $h' = f'$, then if z_0 were to be a non simple zero, then, we must've had:

$$h'(z_0) = 0$$

but this is not the case by the choice of ϵ, therefore, all zeros of h are simple, hence exists at least two distinct $z_0, z_1 \in D_\epsilon(0)$ such that:

$$h(z_0) = 0 \implies f(z_0) = w, \; h(z_1) = 0 \implies f(z_1) = w$$

therefore, f is non-injective as required. $\square$

Lemma 2 if $f\colon \mathbb{C} \to \mathbb{C}$ entire and exists $A, B > 0$ such that:

$$|f(z)| \leq A|z|^k + B$$

then f is a polynomial of degree at most k

Proof. since f is holomorphic at $z = 0$ then there exists $r > 0$ such that:

$$\forall z \in D_r(0) \;\; f(z) = \sum a_n z^n$$

since f is entire, then the radius of convergence must have $R = \infty$. therefore,

$$f(z) = \sum a_n z^n$$

now if we show that $\forall n > k \;\; a_n = 0$, then were done. by cauchy's integral formula we have:

$$a_n = \frac{f^{(n)}}{n!}$$

and by cauchy's theorem we have:

$$|f^{(n)}| \leq \frac{n! \|f\|_{C_r(0)}}{r^n}$$

for $r > 0$, therefore,

$$|a_n| \leq \frac{\|f\|}{r^n}$$

by the assumption, we have:

$$|a_n| \leq \frac{A|z|^k + B}{r^n} = \frac{Ar^k + B}{r^n}$$

fix $n > k$, and take $r \to \infty$, which we can do because f is entire. hence we have:

$$|a_n| \xrightarrow{r \to \infty} 0$$

therefore, $\forall n > k \;\; a_n = 0$, as required. $\square$

Lemma 3 let $p\colon \mathbb{C} \to \mathbb{C}$ be injective polynomial. then p is linear.

Proof. since p is polynomial, then it's of degree m, since p is injective then $m \geq 1$, because if $m = 0$, then it's clearly non injective.

if $m = 1$ were done, hence assume $m \geq 2$, therefore, there exists two roots z_1, z_2 by the fundemental theorem of algebra.

they cannot be distinct, because otherwise p is non injective, since $p(z_1) = p(z_2)$. hence z_1 is a root of order at least 2. therefore:

$$p(z_1) = 0, \; p'(z_1) = 0$$

therefore, by (1.2) p is not injective, which is a contradiction. therefore, p is linear. $\square$

Claim 1 let $f\colon \mathbb{C} \to \mathbb{C}$ be injective entire function, then:

$$f(z) = \alpha z + \beta$$

that is, f is linear.

Proof. define the following function:

$$g(z) = f\left(\frac{1}{z}\right)$$

hence $z = 0$ is a singularity.

First, it cannot be removable, because assume it's removable, then g is bounded in a nghbr $D_\epsilon(0)$.

Hence, take $\epsilon < 1$, then we have, f bounded on $\mathbb{C} \setminus D_{1/\epsilon}(0)$, and by compactness bounded on $\overline{D_{1/\epsilon}}(0)$.

Therefore, f is bounded in $\mathbb{C}$, hence by luiovel theorem it's constant, in contradiction.

Second, $z = 0$ cannot be essential, because assume such, then by corosati-wistraws theorem, $g(D_1'(0))$ is dense, in particular, take:

$$V = \mathbb{C} \setminus \overline{D_1(0)}$$

then,

$$D = f(V)$$

is dense. by the open mapping theorem, we have:

$$U = f(D_1(0))$$

an open set.

hence, for

$$w = f(0) \in U$$

there exists, for $r > 0$:

$$D_r(w) \subset U$$

since D is dense, we must have:

$$D \cap D_r(w) \neq \emptyset$$

which means, there exists, $\alpha \in V,\ \beta \in D_1(0)$ such that:

$$f(\alpha) = f(\beta)$$

but since by definition, we have:

$$V \cap D_1(0) = \emptyset$$

then $\alpha \neq \beta$, which is a contradiction to injectivity.

Therefore, $z = 0$ is a pole.

hence, there exists $h(z)$ holomorphic, and non vanishing on $D_\epsilon(0)$ such that:

$$g(z) = z^{-k} h(z)$$

therefore:

$$f(z) = z^k h\left(\frac{1}{z}\right)$$

hence, for all $z \in \mathbb{C} \setminus D_{1/\epsilon}(0)$ we have:

$$|f(z)| \le |z^k h(1/z)| \le |z^k||h(1/z)|$$

since we have h holomorphic non vanishing on $\overline{D_\delta(0)}$ for $\delta < \epsilon$, then:

$$|h(1/z)| \le K$$

hence we have:

$$|f(z)| \le |z|^k \cdot K$$

for all $R < |z|$ for some $R > 0$. therefore, by (1.2), f is a polynomial of degree at most k, hence it's polynomial.

by (1.3) injective polynomial must be linear as required.

$\square$